Defn: Methods that seek quantitative approximations to the solutions of mathematical problems.

Applications:
* Structural & Stress Analysis
* Thermal analysis
* Dynamic analysis
* Acoustic analysis etc.

Finite Differences.

* First difference:

Let $y = f(x)$ be a given function of x and let $y_0, y_1, y_2, \ldots y_n$ be the values of y corresponding to $x_0, x_1, x_2, \ldots x_n$, the values of x. The independent variable x is called the argument and the corresponding dependent values y is called, the entry.

We can write the arguments and entries as below.

x	x_0	x_1	x_2	$\ldots$	x_{n-1}	x_n
y	y_0	y_1	y_2	$\ldots$	y_{n-1}	y_n

If we subtract from each value of y (except y_0) the preceding value of y, ~~are denoted by~~ $\cancel{\Delta y}$, we get

$y_1 - y_0, \; y_2 - y_1, \; y_3 - y_2, \ldots, y_n - y_{n-1}$

These results are called the first differences of y. The first differences of y are denoted by Δy.

i.e.,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

Here, the symbol Δ denotes an operation, called forward difference operator.

Higher differences:

The second and higher differences are defined as below.

2^{nd} order forward differences of y_0,

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$

$$\vdots$$

$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1}) = \Delta y_n - \Delta y_{n-1}$$

Here, Δ^2 is an operator called, second order forward difference operator.

In the same way, the third order forward difference operator Δ^3 is as follows.

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$

$$\vdots$$

$$\Delta^2 y_{n-1} = \Delta^2 y_n - \Delta^2 y_{n-1}$$

and

$$\Delta^n y_0 = \Delta^{n-1} y_1 - \Delta^{n-1} y_0$$

$$\Delta^n y_1 = \Delta^{n-1} y_2 - \Delta^{n-1} y_1$$

$$\vdots$$

$$\Delta^n y_{n-1} = \Delta^{n-1} y_n - \Delta^{n-1} y_{n-1}$$

In general, we can write

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i .$$

Usually, the arguments are taken as

$$x_0,\ x_0 + h,\ x_0 + 2h,\ x_0 + 3h,\ \dots$$

so that

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots = h$$

Here, h is called the interval of differencing.

$\therefore$ The forward difference operator Δ is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$|||^{rly}$

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x))$$

$$= \Delta\{f(x+2h) - 2f(x+h) + f(x)\}$$

$$= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x)$$

$$= f(x+3h) - f(x+2h) - 2\{f(x+2h) - f(x+h)\} + f(x+h) - f(x)$$

$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$ and so on

<u>Backward difference operator</u> (∇)

Backward Difference Operator (∇) is defined as

$$\nabla f(x) = f(x) - f(x-h).$$

by definition, we have

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1 \text{ etc.}$$

Hence

$$\nabla^2 f(x) = \nabla[f(x) - f(x-h)]$$

$$= \nabla f(x) - \nabla f(x-h)$$

$$= f(x) - f(x-h) - [f(x-h) - f(x-2h)]$$

$$= f(x) - 2f(x-h) + f(x-2h).$$

Central difference Operator (δ)

The central difference operator δ is defined by

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) \quad (\text{or})$$

$$\delta y_x = \cancel{y_x + \frac{h}{2} - y_x - \frac{h}{2}} \qquad y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$$

Shifting or displacement or translation Operator E:

Let the shifting operator E be

$$E f(x) = f(x+h) \quad (\text{or})$$

$$E y_x = y_{x+h}$$

Hence $E y_1 = y_2$

$E y_2 = y_3$ etc.

$$E^2 y_x = E(y_{x+h}) = y_{x+2h}$$

$\vdots$

$$E^n y_x = y_{x+nh} \quad \text{and}$$

$$E^n f(x) = f(x+nh).$$

Inverse Operator (E^{-1}):

Inverse Operator is given by

$$E^{-1} f(x) = f(x-h)$$

$$E^{-2} f(x) = f(x-2h)$$

$\vdots$

$$E^{-n} f(x) = f(x-nh).$$

Averaging Operator (μ):

The averaging operator μ is defined by

$$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right)$$

i.e., $\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$

Differential Operator (D):

The differential operator D is defined by

$$D f(x) = \frac{d}{dx} f(x).$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x). \text{ etc.}$$

Unit Operator 1 :

The unit operator 1 is such that

$$1. f(x) = f(x).$$

Properties of Operators

1. The operators $\Delta, \nabla, E, \delta, \mu$ and D are all linear operators.

Proof: $\Delta(a f(x) + b\,\phi(x)) = [a f(x+h) + b\,\phi(x+h)] - [a f(x) + b\,\phi(x)]$

$$= a[f(x+h) - f(x)] + b[\phi(x+h) - \phi(x)]$$

$$= a\,\Delta f(x) + b\,\Delta\phi(x).$$

Hence Δ is a linear operator.

Result 1 : Put $a=1$ & $b=1$ $\Rightarrow$ $\Delta[f(x)+\phi(x)] = \Delta f(x) - \Delta\phi(x)$.

Result 2 ; Put $b=0 \Rightarrow \Delta(a f(x)) = a\,\Delta f(x)$.

2. The operator is distributive over addition.

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^m \Delta^n f(x).$$

Pf.

$$\Delta^m \Delta^n f(x) = (\Delta\ \Delta \dots m \text{ times})(\Delta\ \Delta \dots n \text{ times}) f(x)$$

$$= \Delta^{m+n} \cdot f(x).$$

$\parallel^{ly}$ We can prove index law for all the other operators.

3. $$\Delta[f(x)+g(x)] = \Delta[g(x)+f(x)] \quad \text{(Commutative law)}$$

Relation between Operators :

(i) Relation between E and $\cancel{\nabla}$ Δ.

We know that,

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - 1. f(x).$$

$$= (E-1) f(x).$$

$$\therefore \boxed{\Delta = E-1} \text{ (or) } \boxed{E = \Delta+1}$$

~~The~~ This ~~operator~~ ~~is~~ is called separation of symbols.

Here, 1 is not the numeral 1 but it is the unit operator 1 which means

$$1. f(x) = f(x).$$

(ii) Relation between E and ∇.

$$\nabla f(x) = f(x) - f(x-h)$$

$$= 1. f(x) - E^{-1} f(x)$$

$$= (1-E^{-1}) f(x).$$

$$\boxed{\nabla = 1-E^{-1}} \text{ (or) } E^{-1} = 1-\nabla \text{ (or) } E = (1-\nabla)^{-1}$$

$$\because (E^{-1})^{-1} = E.$$